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AN INTEGRAL BHATTACHARYYA TYPE BOUND FOR THE BAYES RISK

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ABSTRACT

Bhattacharyya type integral inequalities for the integrated risk for estimators are given extending the work of Borovkov and Sakhanienko (1980). As an application, an asymptotic approximation of the lower bound for locally minimax risk is given.

1. INTRODUCTION

As an application of Cramér-Rao inequality, Borovkov and Sakhanienko (1980) and Brown and Gajek (1990) showed some lower bounds for the Bayes risk under quadratic loss. They also discussed lower bounds for the minimax risk (see also Prakasa Rao (1992), Ghosh (1994), Sato and Akahira (1996) and Koike (1999)).

Unfortunately, these bounds are not always sharp. On the other hand, it is well known that Bhattacharyya type lower bound for the variance of unbiased estimators improves the Cramér-Rao type bound and it converges to the variance of the minimum variance unbiased estimate under some regularity conditions.

The purpose of the paper is to show an extension of Borovkov-Sakhanienko bound for the Bayes risk. As an application, an asymptotic approximation of the lower bound for the local minimax risk is given.

2. A LOWER BOUND FOR THE BAYES RISK

Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed (i.i.d.) random variable according to the density function $f_1(x,t)$ $(t \in \Theta)$ with respect to a σ-finite measure μ, where Θ is an (possibly infinite) interval with the end points a and b $(-\infty \le a < b \le \infty)$. Then the joint probability density function of $X := (X_1, \ldots, X_n)$ is $f(x,t) := \prod_{i=1}^n f_1(x_i,t)$,where $x = (x_1, \ldots, x_n)$. Let $q(t)$ be a prior density of t with respect to Lebesgue measure. Let $\mathrm{supp}(g)$ be the support of a function g on Θ, i.e., $\mathrm{supp}(g) = \{g \neq 0\}$. Consider the problem of Bayes estimation for a thrice differentiable function $g(t)$ of t under quadratic loss $L(t,a) = (a - g(t))^2$.

We make the following conditions.

(A0) For almost all x, $f_1(x,t)$ is twice differentiable with respect to t.

(A1) The 1st and 2nd derivatives with respect to t of the left-hand side of

$$\int_{\mathcal{X}} f(x,t)d\mu = 1$$

can be obtained by differentiating once and twice under the integral sign, respectively, where $\mathcal{X}$ is the sample space of X. And the Fisher information number

$$I(t) = E_t\left[\left\{\frac{\partial}{\partial t}\log f_1(X_1,t)\right\}^2\right] = \int \frac{\{\frac{\partial}{\partial t} f_1(x,t)\}^2}{f_1(x,t)} d\mu$$

exists and $0 < I(t) < \infty$ for arbitrary $t \in \Theta$.

(A2) The prior density q is twice continuously differentiable and $\mathrm{supp}(q) \subset \Theta$.

(A3) $I(t)$ is continuously differentiable and the derivative with respect to t of the left-hand side of the equality of (A1) can be obtained by the differentiating under the integral sign.

Hereafter, we will often omit the variables of the functions. Then we have the following theorem concerning the Bayes risk.

Theorem 1. *Let $\hat{g}(X)$ be an estimator of $g(t)$. Let h be a differentiable function satisfying $\mathrm{supp}(h) \subset \mathrm{supp}(q)$. Suppose that, for almost all x, $h(t)f_1(x,t) = \frac{\partial}{\partial t}\{h(t)f_1(x,t)\} = 0$ at $t = a$ and b. Then, under the conditions (A0)–(A2), it holds*

$$B\left(\hat{g}, q\right) \ge \left(E\left(\frac{g'h}{q}\right), -E\left(\frac{g''h}{q}\right)\right) V^{-1} \left(E\left(\frac{g'h}{q}\right), -E\left(\frac{g''h}{q}\right)\right)', \tag{2.1}$$

where $V = \{E(S_iS_j)\}_{i,j=1,2}$ *is a* 2×2 *matrix with*

$$
\begin{aligned}
E\left(S_1^2\right) =& nE\left(\frac{h^2 I}{q^2}\right) + E\left\{\left(\frac{h'}{q}\right)^2\right\}, \\
E\left(S_1S_2\right) =& n\left[E\left\{\left(\frac{h}{q}\right)^2 E_t\left(\frac{f_1'f_1''}{f_1^2}\right)\right\} + 2E\left(\frac{Ihh'}{q^2}\right)\right] + E\left(\frac{h'h''}{q^2}\right), \\
E\left(S_2^2\right) =& 2n^2 E\left\{\left(\frac{Ih}{q}\right)^2\right\} + n\left[E\left\{\left(\frac{h}{q}\right)^2 E_t\left(\frac{f_1''}{f_1}\right)^2\right\} - 2E\left\{\left(\frac{Ih}{q}\right)^2\right\}\right. \\
& \left. +4E\left(\frac{Ih'^2}{q^2}\right) + 4E\left\{\frac{hh'}{q^2}E_t\left(\frac{f_1'f_1''}{f_1^2}\right)\right\}\right] + E\left\{\left(\frac{h''}{q}\right)^2\right\}.
\end{aligned}
$$

Proof. Let $S_i = \{f(x,t)q(t)\}^{-1}(\partial^i/\partial t^i)\{f(x,t)h(t)\}$ $(i = 1,2)$. Considering the covariance matrix U of the random vector $(\hat{g} - g, S_1, S_2)$, we can show that U is a symmetric matrix given by

$$
U = \begin{pmatrix} E\{(\hat{g}-g)^2\} & E(g'h/q) & -E(g''h/q) \\ E(g'h/q) & E(S_1^2) & E(S_1S_2) \\ -E(g''h/q) & E(S_1S_2) & E(S_2^2) \end{pmatrix}. \tag{2.2}
$$

Indeed, integrating by parts, we have

$$
\begin{aligned}
&\int_\Theta \frac{\partial}{\partial t}\{f(x,t)h(t)\}\,dt = [f(x,t)h(t)]_a^b = 0, \\
&\int_\Theta g(t)\frac{\partial}{\partial t}\{f(x,t)h(t)\}\,dt = -\int_\Theta g'(t)f(x,t)h(t)dt
\end{aligned}
$$

from $h(a) = h(b) = 0$. Then, we have

$$
\begin{aligned}
E\{(\hat{g}-g)S_1\} &= \int_\mathcal{X}\int_\Theta (\hat{g}-g)\frac{\partial}{\partial t}\{f(x,t)h(t)\}\,dtd\mu \\
&= \int_\mathcal{X} \hat{g}\int_\Theta \frac{\partial}{\partial t}\{f(x,t)h(t)\}\,dtd\mu - \int_\mathcal{X}\int_\Theta g\frac{\partial}{\partial t}\{f(x,t)h(t)\}\,dtd\mu \\
&= \int_\mathcal{X}\int_\Theta g'(t)f(x,t)h(t)dtd\mu \\
&= \int_\Theta g'(t)h(t)\int_\mathcal{X} f(x,t)d\mu dt \\
&= \int_\Theta g'(t)h(t)dt = E\left(\frac{g'h}{q}\right).
\end{aligned} \tag{2.3}
$$

Similarly, since $(\partial/\partial t)\{f(x,t)h(t)\}=0$ for $t=a$ and b, we have

$$
\begin{aligned}
& E\{(\hat{g}-g)S_2\} \\
&= \int_{\mathcal{X}}\int_{\Theta}(\hat{g}-g)\frac{\partial^2}{\partial t^2}\{f(x,t)h(t)\}\,dtd\mu \\
&= \int_{\mathcal{X}}\hat{g}\int_{\Theta}\frac{\partial^2}{\partial t^2}\{f(x,t)h(t)\}\,dtd\mu - \int_{\mathcal{X}}\int_{\Theta} g\frac{\partial^2}{\partial t^2}\{f(x,t)h(t)\}\,dtd\mu \\
&= \int_{\mathcal{X}}\hat{g}\left[\frac{\partial}{\partial t}\{f(x,t)h(t)\}\right]_a^b d\mu + \int_{\mathcal{X}}\int_{\Theta} g'\frac{\partial}{\partial t}\{f(x,t)h(t)\}dtd\mu \\
&= -\int_{\mathcal{X}}\int_{\Theta} g''(t)f(x,t)h(t)d\mu dt \\
&= -\int_{\Theta} g''(t)h(t)\int_{\mathcal{X}} f(x,t)d\mu dt \\
&= -\int_{\Theta} g''hdt = -E\left(\frac{g''h}{q}\right).
\end{aligned} \tag{2.4}
$$

On the other hand, from Borovkov and Sakhanienko (1980),

$$
E(S_1^2) = nE\left(\frac{h^2I}{q^2}\right) + E\left\{\left(\frac{h'}{q}\right)^2\right\}. \tag{2.5}
$$

Define $L'=\{f(x,t)\}^{-1}(\partial/\partial t)f(x,t)$ and $L''=\{f(x,t)\}^{-1}(\partial^2/\partial^2 t)f(x,t)$. By the condition (A3), it holds $E_t(L')=E_t(L'')=0$ (see also Borovkov (1998)) so that

$$
\begin{aligned}
E(S_1S_2) &= E\left\{\left(L'\frac{h}{q}+\frac{h'}{q}\right)\left(L''\frac{h}{q}+2L'\frac{h'}{q}+\frac{h''}{q}\right)\right\} \\
&= E\left\{L'L''\left(\frac{h}{q}\right)^2 + 2L'^2\frac{hh'}{q^2} + L'\frac{hh''}{q^2} + L''\frac{hh'}{q^2} + 2L'\left(\frac{h'}{q^2}\right)^2 + \frac{h'h''}{q^2}\right\} \\
&= E\left\{L'L''\left(\frac{h}{q}\right)^2 + 2L'^2\frac{hh'}{q^2} + \frac{h'h''}{q^2}\right\},
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
E\left(S_2^2\right) &= E\left\{\left(L''\frac{h}{q}+2L'\frac{h'}{q}+\frac{h''}{q}\right)^2\right\} \\
&= E\left\{L''^2\left(\frac{h}{q}\right)^2 + 4L'^2\left(\frac{h'}{q}\right)^2 + 4L'L''\frac{hh'}{q^2} + 2L''\frac{hh''}{q^2} + 4L'\frac{h'h''}{q^2} + \left(\frac{h''}{q}\right)^2\right\}.
\end{aligned} \tag{2.7}
$$

We have from the definition of L' and L''

$$L' = \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(x_i, t), \quad L'' = \sum_{i=1}^{n} \frac{\partial^2}{\partial t^2} \log f_1(x_i, t) + \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(x_i, t) \right\}^2,$$

so that

$$E_t(L'^2) = E_t \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(X_i, t) \right\}^2 = nI(t), \tag{2.8}$$

$$\begin{aligned}
E_t(L'L'') &= E_t \left[\left\{ \sum_{i=1}^{n} \frac{\partial^2}{\partial t^2} \log f_1(X_i, t) \right\} \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(X_i, t) \right\} + \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(X_i, t) \right\}^3 \right] \\
&= nE_t \left[\left\{ \frac{\partial^2}{\partial t^2} \log f_1(X_i, t) \right\} \left\{ \frac{\partial}{\partial t} \log f_1(X_i, t) \right\} \right] + nE_t \left[\left\{ \frac{\partial}{\partial t} \log f_1(X_i, t) \right\}^3 \right] \\
&= nE_t \left\{ \frac{f_1' f_1''}{f_1^2} - \left(\frac{f_1'}{f_1^2} \right)^3 \right\} + nE_t \left\{ \left(\frac{f_1'}{f_1} \right)^3 \right\} \\
&= nE_t \left(\frac{f_1' f_1''}{f_1^2} \right),
\end{aligned} \tag{2.9}$$

where $f_1' = \partial f(X_i, t)/\partial t$ and $f_1'' = \partial^2 f(X_i, t)/\partial t^2$.

In a similar way to the above, we have for $E(L''^2)$

$$\begin{aligned}
E_t\left(L''^2\right) &= E_t \left[\left\{ \sum_{i=1}^{n} \frac{\partial^2}{\partial t^2} \log f_1(X_i, t) \right\}^2 + \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(X_i, t) \right\}^4 \right. \\
&\quad \left. +2 \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial t} \log f_1(X_i, t) \right\}^2 \left\{ \sum_{i=1}^{n} \frac{\partial^2}{\partial t^2} \log f_1(X_i, t) \right\} \right] \\
&= nE_t \left\{ \left(\frac{f_1''}{f_1} \right)^2 - 2\frac{f_1' f_1''}{f_1} + \left(\frac{f_1'}{f_1} \right)^4 \right\} + nE_t \left\{ \left(\frac{f_1'}{f_1} \right)^4 \right\} + 3n(n-1)I^2 \\
&\quad + 2 \left[nE_t \left\{ \frac{f_1'^2 f_1''}{f_1} - \left(\frac{f_1'}{f_1} \right)^4 \right\} - n(n-1)I^2 \right].
\end{aligned} \tag{2.10}$$

Then, from (2.8), (2.9) and (2.10), the right-hand sides of (2.6) and (2.7) are equal to

$$E(S_1S_2) = nE\left\{E_t\left(\frac{f_1'f_1''}{f_1^2}\right)\left(\frac{h}{q}\right)^2 + 2I\frac{hh'}{q^2}\right\} + E\left\{\frac{h'h''}{q^2}\right\}, \tag{2.11}$$

$$\begin{aligned} E\left(S_2^2\right) = 2n^2E\left\{\left(\frac{Ih}{q}\right)^2\right\} &+ nE\left[\left(\frac{h}{q}\right)^2 E_t\left\{\left(\frac{f_1''}{f_1}\right)^2\right\}\right. \\ &\left. -2\left(\frac{Ih}{q}\right)^2 + 4I\left(\frac{h'}{q}\right)^2 + 4E_t\left(\frac{f_1'f_1''}{f_1^2}\right)\frac{hh'}{q^2}\right] + E\left\{\left(\frac{h''}{q}\right)^2\right\}. \end{aligned} \tag{2.12}$$

On the other hand, since U is nonnegative definite, it follows that

$$|U| = |V|\left|E\{(\hat{g}-g)^2\} - \left(E(g'h/q), -E(g''h/q)\right)V^{-1}\left(E(g'h/q), -E(g''h/q)\right)'\right| > 0,$$

where $V = \{E(S_iS_j)\}_{i,j=1,2}$. And then, we have

$$B(\hat{g},q) = E\{(\hat{g}-g)^2\} \geq \left(E(g'h/q), -E(g''h/q)\right)V^{-1}\left(E(h/q), -E(g''h/q)\right)' \tag{2.13}$$

Therefore, from (2.3), (2.4), (2.5), (2.11), (2.12) and (2.13), we have the desired inequality.

Choosing $h = g'q/I$, we have the following corollary.

Corollary. *If* $h = g'q/I$ *is differentiable and* $\mathrm{supp}(h) \subset \mathrm{supp}(q)$, *then we have, under the conditions (A0)–(A2),*

$$B\left(\hat{g},q\right) \geq \left(\int_\Theta \frac{g'^2q}{I}dt, -\int_\Theta \frac{g'g''q}{I}dt\right)\tilde{V}^{-1}\left(\int_\Theta \frac{g'^2q}{I}dt, -\int_\Theta \frac{g'g''q}{I}dt\right)', \tag{2.14}$$

where $\tilde{V} = \{E(S_iS_j)\}_{i,j=1,2}$ *is a* 2×2 *matrix with*

$$E\left(S_1^2\right) = n\int_\Theta \frac{g'^2q}{I}dt + \int_\Theta \frac{1}{q}\left(g''\frac{q}{I} + g'\left(\frac{q}{I}\right)'\right)^2 dt,$$

$$\begin{aligned} E(S_1S_2) = n&\left[\int_\Theta \frac{g'^2q}{I^2}\int_{\mathcal{X}}\frac{f_1'f_1''}{f_1}d\mu dt - 2\int_\Theta \frac{g'g''q}{I}dt\right] \\ &+ \int_\Theta \frac{1}{q}\left(\frac{g''q}{I} + g'\left(\frac{q}{I}\right)'\right)\left(\frac{g'''q}{I} + 2g''\left(\frac{q}{I}\right)' + g'\left(\frac{q}{I}\right)''\right)dt, \end{aligned}$$

$$\begin{aligned} E(S_2^2) = 2n^2\int_\Theta g'^2q dt + n&\left[\int_\Theta \frac{g'^2q}{I^2}\int_{\mathcal{X}}\frac{f_1''^2}{f_1}d\mu - 2\int_\Theta g'^2 q dt\right. \\ &\left. + 4\int_\Theta \frac{I}{q}\left(\frac{g''q}{I} + g'\left(\frac{q}{I}\right)'\right)^2 dt + 4\int_\Theta \frac{g'}{I}\left(\frac{g''q}{I} + g'\left(\frac{q}{I}\right)'\right)\int_{\mathcal{X}}\frac{f_1'f_1''}{f_1}d\mu dt\right] \\ &+ \int_\Theta \frac{1}{q}\left(\frac{g'''q}{I} + 2g''\left(\frac{q}{I}\right)' + g'\left(\frac{q}{I}\right)''\right)^2 dt, \end{aligned}$$

and the asymptotic approximation of (2.14) is

$$\left(\int_\Theta \frac{g'^2 q}{I} dt\right) n^{-1} + \left\{ - \int_\Theta \left(\frac{g'' q}{I} + g' \left(\frac{q}{I}\right)' \right)^2 \frac{dt}{q} \right.$$
$$\left. + \frac{1}{2\int_\Theta g'^2 q dt} \left(\int_\Theta \frac{g'^2 q}{I^2} \int_\mathcal{X} \frac{f_1' f_1''}{f_1} d\mu dt - \int_\Theta \frac{g' g'' q}{I} dt \right)^2 \right\} n^{-2} + O\left(n^{-3}\right) \tag{2.15}$$

as $n \to \infty$.

Proof. By substituting $h = g'q/I$ in (2.1), we have the first inequality. For the second inequality, the asymptotic approximation of the right-hand side of (2.14) is given by

$$\left(\int_\Theta \frac{g'^2 q}{I} dt\right) n^{-1} + \left\{ - \int_\Theta \frac{h'^2}{q} dt + \frac{1}{2\int_\Theta g'^2 q dt} \left(\int_\Theta \frac{g'^2 q}{I^2} \int_\mathcal{X} \frac{f_1' f_1''}{f_1} d\mu dt + 2 \int_\Theta \frac{g' h'}{q} dt \right)^2 \right.$$
$$\left. + \frac{\int_\Theta \frac{g' g'' q}{I} dt}{2\int_\Theta g'^2 q dt} \left(2 \int_\Theta \frac{g'^2 q}{I^2} \int_\mathcal{X} \frac{f_1' f_1''}{f_1} d\mu dt + 4 \int_\Theta g' h' dt + \int_\Theta \frac{g' g'' q}{I} dt \right) \right\} n^{-2} + O\left(n^{-3}\right).$$

Since

$$\int_\Theta g' h' dt = - \int_\Theta \frac{g' g'' q}{I} dt$$

by using the integration by parts, we have the desired results.

Remark. (1) In particular, substituting $g = t$, the bound (2.15) equals

$$\left(\int_\Theta \frac{q}{I} dt\right) n^{-1} + \left\{ - \int_\Theta \frac{1}{q} \left(\frac{q}{I}\right)'^2 dt + \frac{1}{2} \left(\int_\Theta \frac{q}{I^2} \int_\mathcal{X} \frac{f_1' f_1''}{f_1} d\mu dt \right)^2 \right\} n^{-2} + O\left(n^{-3}\right)$$

as $n \to \infty$.

(2) Applying a similar approximation to (2.15) for the Borovkov-Sakhanienko inequality (1980), we have

$$\begin{aligned}
B\left(\hat{g}, q\right) \geq & \frac{\left(\int_\Theta \frac{g'^2 q}{I} dt\right)^2}{n \int_\Theta \frac{g'^2 q}{I} dt + \int_\Theta \frac{h'^2}{q} dt} \\
= & \left(\int_\Theta \frac{g'^2 q}{I} dt\right) n^{-1} + \left(- \int_\Theta \frac{h'^2}{q} dt \right) n^{-2} + O\left(n^{-3}\right) \\
= & \left(\int_\Theta \frac{g'^2 q}{I} dt\right) n^{-1} + \left\{ - \int_\Theta \frac{1}{q} \left(\frac{g'' q}{I} + g' \left(\frac{q}{I}\right)' \right)^2 dt \right\} n^{-2} + O\left(n^{-3}\right)
\end{aligned} \tag{2.16}$$

as $n \to \infty$, where $h = g'q/I$. So, the coefficient of n^{-1} for (2.15) coincides with the one of (2.16) and the difference between the bounds (2.15) and (2.16) up to the order of n^{-2} is

$$\frac{1}{2\int_\Theta g'^2 q dt}\left(\int_\Theta \frac{g'^2 q}{I^2}\int_{\mathcal{X}} \frac{f_1' f_1''}{f_1} d\mu dt - \int_\Theta \frac{g' g'' q}{I} dt\right)^2 n^{-2} \geq 0.$$

3. EXAMPLES

In this section, we will show some examples.

Example 3.1. Let $X_1, \ldots, X_n$ be i.i.d. as $N(t, 1)$, the normal distribution with mean t and variance 1. Consider the Bayes estimation of $g(t) = t^2$ under quadratic loss when the prior distribution of t is $N(\mu, \sigma^2)$. The posterior density of t given $X_1, \ldots, X_n$ is $N\left(\frac{\sum_{i=1}^n X_i + (\mu/\sigma^2)}{n+(1/\sigma^2)}, \frac{1}{n+(1/\sigma^2)}\right)$. Thus the Bayes estimator of $g(t) = t^2$ is

$$\hat{g} = E\left(t^2 | X_1, \ldots, X_n\right) = \frac{1}{n + (1/\sigma^2)} + \left\{\frac{\sum_{i=1}^n X_i + (\mu/\sigma^2)}{n + (1/\sigma^2)}\right\}^2.$$

An easy computation yields

$$\begin{aligned} E\left\{(\hat{g} - t^2)^2\right\} =& \frac{2\sigma^2(2n\sigma^4 + 2\mu^2\sigma^2 n + 2\mu^2 + \sigma^2)}{(n\sigma^2 + 1)^2} \\ =& 4(\mu^2 + \sigma^2)n^{-1} + \frac{-2(2\mu^2 + 3\sigma^2)}{\sigma^2} n^{-2} + O\left(n^{-3}\right) \quad (n \to \infty). \end{aligned}$$

On the other hand, since $g'(t) = 2t$, $I(t) = E_t\left\{(\frac{\partial}{\partial t}\log f_1)^2\right\} = 1$ and $\int \frac{f_1' f_1''}{f_1} dx = 0$, the right-hand sides of (2.15) and (2.16) are

$$\begin{aligned} &(2.15): \quad 4(\mu^2 + \sigma^2)n^{-1} + \left\{-4\left(2 + \frac{\mu^2}{\sigma^2}\right) + \frac{2\mu^2}{(\mu^2 + \sigma^2)}\right\} n^{-2} + O\left(n^{-3}\right), \\ &(2.16): \quad 4(\mu^2 + \sigma^2)n^{-1} - 4\left(2 + \frac{\mu^2}{\sigma^2}\right) n^{-2} + O\left(n^{-3}\right) \end{aligned}$$

as $n \to \infty$, respectively. Then the difference between the Bayes risk of $\hat{g}$ and (2.15) is

$$\left(2 - \frac{2\mu^2}{\mu^2 + \sigma^2}\right) n^{-2} + O\left(n^{-3}\right) \geq 0 \quad (n \to \infty).$$

The coefficient of n^{-2} tends to 0 as $|\mu| \to \infty$ or $\sigma^2 \to 0$. But the difference between the Bayes risk of $\hat{g}$ and (2.16) is $2n^{-2} + O\left(n^{-3}\right)$ $(n \to \infty)$ and the infimum of the coefficient of n^{-2} is still 2.

Example 3.2. Let $X_1, \ldots, X_n$ be i.i.d. according to the density function

$$f_1(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & (x > 0), \\ 0 & (\text{otherwise}), \end{cases}$$

where $\lambda > 0$ is an unknown parameter from a given parameter set $\Theta = (0, \infty)$. Let q be the prior density of λ given by

$$q(\lambda; p, a) = \begin{cases} \frac{a^p}{\Gamma(p)} \lambda^{p-1} e^{-a\lambda} & (\lambda > 0), \\ 0 & (\lambda \leq 0) \end{cases}$$

where $a, p > 0$. It is well known that $T_n = n^{-1} \sum_{i=1}^n X_i$ is sufficient for λ and its density function is given by

$$f_{T_n}(x;\lambda) = \begin{cases} \frac{(\lambda n)^n}{\Gamma(n)} x^{n-1} e^{-n\lambda x} & (x > 0), \\ 0 & (\text{otherwise}). \end{cases}$$

We consider the estimation of reliability function of the form

$$R(c) = P(X_1 \geq c) = e^{-c\lambda} \quad (c > 0).$$

The Bayes estimator of $R(c)$, which is obtained as the expectation of $e^{-c\lambda}$ with respect to the posterior distribution, is

$$\hat{R} = \left(\frac{nT_n + a}{nT_n + a + c} \right)^{n+p}$$

(see Antoch et al. (1997)). The direct calculation of the Bayes risk of $\hat{R}$ is difficult and Antoch et al. (1997) derived the asymptotic approximation of it. Here we will show some comparison between the Bayes risk and the lower bound from the asymptotic point of view.

Put $g(\lambda) = e^{-c\lambda}$. By a simple calculation, we have $g'(\lambda) = -ce^{-c\lambda}$,
$I = -E_\lambda\{\frac{\partial^2}{\partial \lambda^2} \log f_1(X_1, \lambda)\} = 1/\lambda^2$,

$$\int_0^\infty \frac{g'^2 q}{I} d\lambda = \frac{a^p c^2 p(p+1)}{(a+2c)^{p+2}}, \quad \int_0^\infty \frac{g'^2 q}{I^2} \int_0^\infty \frac{f_1' f_1''}{f_1} d\mu d\lambda = -2 \frac{a^p c^2 p}{(a+2c)^{p+1}},$$

$$\int_0^\infty \left(\frac{g'' q}{I} + g' \left(\frac{q}{I} \right)' \right)^2 \frac{d\lambda}{q} = \frac{a^p c^2 p(p+1)}{(a+2c)^{p+2}} \left(3a^2 + 4ac + 2c^2 + p^2 c^2 + pa^2 + pc^2 \right),$$

$$\int_0^\infty g'^2 q d\lambda = \frac{a^p c^2}{(a+2c)^p}, \quad \int_0^\infty \frac{g' g'' q}{I} d\lambda = -\frac{a^p c^3 p(p+1)}{(a+2c)^{p+2}}.$$

So, the asymptotic approximations of the lower bounds (2.15) and (2.16) are given by

$$\begin{aligned}
(2.15):\quad & \frac{a^p c^2 p(p+1)}{(a+2c)^{p+2}} n^{-1} + \frac{a^p c^2 p(p+1)}{2(a+2c)^{p+4}} \{ -c^2p^2 - p(a^2+c^2) \\
& - 3a^2 - 2c^2 - 4ac \} n^{-2} + \frac{a^p c^2 p^2}{2(a+2c)^{p+4}} (-2a - 3c + cp)^2 n^{-2} + O\left(n^{-3}\right) \\
= & \frac{a^p c^2 p(p+1)}{(a+2c)^{p+2}} n^{-1} + \frac{a^p c^2 p}{2(a+2c)^{p+4}} \{ -c^2p^3 - 2p^2(a^2+5c^2+2ac) \\
& + p(-4a^2+3c^2+4ac) - 6a^2 - 8ac - 4c^2 \} n^{-2} + O\left(n^{-3}\right), \qquad (3.1) \\
(2.16):\quad & \frac{a^p c^2 p(p+1)}{(a+2c)^{p+2}} n^{-1} + \frac{a^p c^2 p(p+1)}{2(a+2c)^{p+4}} \{ -c^2p^2 - p(a^2+c^2) \\
& - 3a^2 - 2c^2 - 4ac \} n^{-2} + O\left(n^{-3}\right),
\end{aligned}$$

respectively. On the other hand, the asymptotic approximation of the Bayes risk is given by

$$\frac{a^p c^2 p(p+1)}{(a+2c)^{p+2}} n^{-1} + \frac{a^p c^2 p(p+1)}{2(a+2c)^{p+4}} \{-c^2p^2 - p(2a^2+5c^2+4ac) - 2a^2 + 2c^2\} n^{-2} + O(n^{-3}) \qquad (3.2)$$

as $n \to \infty$ (see Antoch et al. (1997)). The difference between the coefficients of n^{-2} for (3.1) and (3.2) is

$$\frac{a^p c^2 p}{2(a+2c)^{p+4}} \{4(a - cp + c)^2 + 2c^2 p + 2c^2\} > 0$$

for all $a, c, p > 0$. Therefore the bound (3.1) improves (2.18), but is not attained by the Bayes risk (3.2) of $\hat{R}$.

4. A LOWER BOUND FOR THE LOCAL MINIMAX RISK

In this section, we consider the efficiency for the minimax estimation of t. Under the conditions of Theorem 1, define $j(t) := E_t \left(-3 \frac{f_1' f_1''}{f_1^2} + 2 \left(\frac{f_1'}{f_1^3} \right)^3 \right)$. Then, we have the following lower bound for the local minimax risk.

Theorem 2. *Suppose that there exists an $\varepsilon > 0$ and $a > 0$ satisfying*

$$0 < a \le j(t) \quad \textit{or} \quad j(t) < -a \le 0 \qquad (4.1)$$

for $t \in (t_0-\varepsilon, t_0+\varepsilon)$. If $(t_0-\varepsilon, t_0+\varepsilon) \subset \Theta$, then we have for the local minimax risk at t_0, under the conditions (A0)–(A3),

$$\sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} E_t\left\{(\hat{t}-t)^2\right\} \geq \frac{1}{I^*}n^{-1} - \frac{\pi^2}{\varepsilon^2 I_*^2}n^{-2} + \frac{a^2}{2{I^*}^3} + O\left(n^{-3}\right) \quad (n\to\infty) \tag{4.2}$$

for any estimator $\hat{t}$ of t, where $I^* = \sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} I(t)$ and $I_* = \inf_{t\in(t_0-\varepsilon,t_0+\varepsilon)} I(t)$.

Proof. Putting $g=t$ and $h=q$ into (2.1), we have

$$\begin{aligned} B(\hat{t},q) \geq & \frac{1}{E(I)}n^{-1} - \frac{E(q'/q)^2}{(E(I))^2}n^{-2} + \frac{1}{2\left(E(I)\right)^2 E(I^2)}\left\{E\left(E_t\left(\frac{f_1'f_1''}{f_1^2}\right)\right) + 2E\left(\frac{Iq'}{q}\right)\right\}^2 n^{-2} \\ & + O(n^{-3}) \quad (n\to\infty) \end{aligned} \tag{4.3}$$

for all $\hat{t}$ and q satisfying the conditions of Theorem 1. Under the condition (A3), since

$$\frac{\partial}{\partial t}I(t) = \frac{\partial}{\partial t}\int \frac{{f_1'}^2}{f_1}d\mu = \int \frac{\partial}{\partial t}\left(\frac{{f_1'}^2}{f_1}\right)d\mu = 2\int \frac{f_1'f_1''}{f_1}d\mu - \int \frac{{f_1'}^3}{f_1^2}d\mu,$$

integrating by parts gives

$$E\left(\frac{Iq'}{q}\right) = -2E\left(E_t\left(\frac{f_1'f_1''}{f_1^2}\right)\right) + E\left(E_t\left(\frac{f_1'}{f_1}\right)^3\right).$$

So, the right-hand side of (4.3) is equal to

$$\begin{aligned} & \frac{1}{E(I)}n^{-1} - \frac{E(q'/q)^2}{(E(I))^2}n^{-2} + \frac{1}{2\left(E(I)\right)^2 E(I^2)}\left\{E\left(E_t\left(-3\frac{f_1'f_1''}{f_1^2} + 2\left(\frac{f_1'}{f_1}\right)^3\right)\right)\right\}^2 n^{-2} \\ & \qquad + O(n^{-3}) \\ \geq & \frac{1}{I^*}n^{-1} - \frac{E(q'/q)^2}{I_*^2}n^{-2} + \frac{a^2}{2{I^*}^3}n^{-2} + O(n^{-3}) \quad (n\to\infty) \end{aligned} \tag{4.4}$$

If we put $q(t) = \frac{1}{\varepsilon}\cos^2\frac{\pi(t-a)}{2\varepsilon}$ for $|t-t_0| \leq \varepsilon$, the right-hand side of (4.4) is

$$\frac{1}{I^*}n^{-1} - \frac{\pi^2}{\varepsilon^2 I_*^2}n^{-2} + \frac{a^2}{2{I^*}^3}n^{-2} + O(n^{-3}) \quad (n\to\infty)$$

Note that

$$\sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} E_t\left\{(\hat{t}-t)^2\right\} \geq B(\hat{t},q),$$

where the expectation of the right-hand side of the above is taken by a density q satisfying $\mathrm{supp}(q) \subset (t_0 - \varepsilon, t_0 + \varepsilon)$. Therefore we have the desired result.

Remark. (1) Note that $q(t) = \frac{1}{\varepsilon}\cos^2\frac{\pi(t-a)}{2\varepsilon}$ attains the minimum of the functional $\int \frac{q'^2}{q} dt$ (see Ghosh (1994) and Borovkov (1998)).
(2) The condition (4.1) is satisfied if $j(t)$ is continuous, $j(t_0) \neq 0$ and $\varepsilon > 0$ is sufficiently small.
(3) Borovkov (1998) gives a similar lower bound:

$$\sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} E_t\left\{(\hat{t}-t)^2\right\} \geq \frac{1}{nE(I)+\pi^2/\varepsilon^2}$$

for any estimator $\hat{t}$ of t, where the expectation $E(\cdot)$ of the right-hand side's denominator is taken by a density q satisfying $\mathrm{supp}(q) \subset (t_0 - \varepsilon, t_0 + \varepsilon)$. This means

$$\sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} E_t\left\{(\hat{t}-t)^2\right\} \geq \frac{1}{I^*}n^{-1} - \frac{\pi^2}{\varepsilon^2 I_*^2}n^{-2} + O\left(n^{-3}\right) \quad (n\to\infty). \tag{4.5}$$

Thus the lower bound (4.2) improves (4.5) up to the order of n^{-2}.

Let $\exp\{a(t)T(x) - \gamma(t)\}$ be the density function of X given t with respect to a σ-finite measure μ, where $a(t)$ is a thrice differentiable monotone function of t and $a'(t) \neq 0$. Then easy computation yields $j(t) = \frac{a'''(t)}{a'(t)}\gamma'(t) - \gamma'''(t)$.

If the assumption (4.1) is not satisfied, (4.2) can be replaced by

$$\sup_{t\in(t_0-\varepsilon,t_0+\varepsilon)} E_t\left\{(\hat{t}-t)^2\right\} \geq \frac{1}{I^*}n^{-1} - \frac{\pi^2}{\varepsilon^2 I_*^2}n^{-2} + O\left(n^{-3}\right) \quad (n\to\infty),$$

from the left-hand side of (4.4). But this lower bound is equal to (4.4). For example, let X be i.i.d. as $N(t,1)$. Since $a(t) = t$ and $\gamma(t) = t^2/2$, we have $j(t) = 0$.

5. SUMMARY

A lower bound for the Bayes risk was obtained. The obtained bound improves the Borovkov-Sakhanienko bound and the asymptotic expression was proved. As an application of the bound, a lower bound for the minimax risk was given.

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